

Overcoming Nonrenormalizability. Part 2

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The procedures to overcome nonrenormalizability of φ_n^4 , $n \geq 5$, quantum field theory models that were presented in a recent paper are extended to address nonrenormalizability of φ_p^p , $p = 8, 10, 12, \dots$, models. The principles involved in these procedures are based on the hard-core picture of nonrenormalizability.

KEY WORDS: Quantum field theory; nonrenormalizability.

INTRODUCTION

The present paper may be regarded as an addendum to a recent paper,⁽¹⁾ where proposals were advanced to overcome nonrenormalizability for φ_n^4 models, quartic self-interacting scalar field models with spacetime dimension $n \geq 5$. (Similar techniques were also proposed to overcome triviality for the renormalizable but not asymptotically free model φ_4^4 .) In this short note we extend the same scheme to advance a proposal designed to overcome nonrenormalizability in models such as φ_3^p , for $n = 3$ and powers $p = 8, 10, 12, \dots$, as well. (If the renormalizable model φ_3^6 is trivial and not asymptotically free, then our proposals should work for this model as well.) For background as well as notational questions we urge the reader to consult.⁽¹⁾

The philosophy underlying our formulation is based on the hard-core picture of nonrenormalizable interactions. A brief introduction to this viewpoint is offered in ref. 1; a more detailed discussion appears in ref. 2. As one important consequence we are led to consider renormalization counterterms that are entirely different from those suggested by conventional perturbation analyses. Let us start with an heuristic, motivational discussion.

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We work entirely in Euclidean spacetime and assume the theories of interest arise as suitable continuum and infinite volume limits of a lattice model formulated on a large but finite cubic lattice with a dimensionless lattice spacing a and periodic boundary conditions. From the viewpoint of critical phenomena the models in question involve multicritical points, and therefore upper critical dimensions, above which mean field arguments are generally applicable, depend on the choice of p . It is straightforward to show that several correlation functions of interest for φ_3^p models are given as follows (see, e.g., refs. 3–5):

$$\Sigma_k \langle \varphi_0 \varphi_k \rangle \propto a^{-2},$$

$$\Sigma_k k^2 \langle \varphi_0 \varphi_k \rangle \propto a^{-4},$$

$$\Sigma_{k_2, k_3, \dots, k_{2r}} \langle \varphi_0 \varphi_{k_2} \varphi_{k_3} \cdots \varphi_{k_{2r}} \rangle^T \propto a^{[2p-4r(p-1)]/[p-2]},$$

for relevant r values of the form $r = 1 + j(p-2)/2$, $j = 0, 1, 2, \dots$, and where $k \in \mathbb{Z}^3$ denotes a lattice site, T denotes the truncated (or connected) component, and we have assumed all odd-order correlation functions vanish. It is conventional to recast these expressions into the single combination

$$g_r \equiv - \frac{\Sigma_{k_2, k_3, \dots, k_{2r}} \langle \varphi_0 \varphi_{k_2} \varphi_{k_3} \cdots \varphi_{k_{2r}} \rangle^T}{[\Sigma_k \langle \varphi_0 \varphi_k \rangle]^r [\Sigma_k k^2 \langle \varphi_0 \varphi_k \rangle / 6 \Sigma_k \langle \varphi_0 \varphi_k \rangle]^{3(r-1)/2}}.$$

The expression for g_r is dimensionless, enjoys rescaling invariance (i.e., $\varphi_k \rightarrow S\varphi_k$, $S > 0$, for all k), and admits a meaningful continuum limit. In particular, for small a it follows that

$$g_r \propto a^{(r-1)(p-6)/(p-2)}.$$

Therefore, when $p \geq 8$ and $a \rightarrow 0$ we find that $g_r \rightarrow 0$ for all relevant $r \geq p/2$. This behavior—which is analogous to what one finds for φ_n^4 models when $n \geq 5$ ⁽⁶⁾—strongly suggests, in the continuum limit, that the nonrenormalizable φ_3^p models exhibit “infidelity.” By infidelity we mean that the resultant quantum theory has a trivial classical limit, clearly differing from the original classical theory, and thus casting doubt on the quantization procedure itself. This result arises because (i) the quantization loses the ϕ_3^p interaction (a conclusion supported by renormalization group analysis), and (ii) any surviving interactions, e.g., ϕ_3^4 and possibly ϕ_3^6 , have been induced and therefore arise from one or more loop contributions. Such terms therefore have \hbar -dependent coupling constants. Thus, in the classical limit where $\hbar \rightarrow 0$ all interactions disappear leading to classical triviality.

The dependence on the lattice spacing a that has led to this claim has arisen from summing over the whole lattice and is based on divergent behavior that emerges near a second-order phase transition. If, by some procedure, we could *simultaneously* rescale all correlation functions *uniformly* so that

$$\langle \varphi_{k_1} \varphi_{k_2} \cdots \varphi_{k_{2r}} \rangle \propto a^{(p-6)/(p-2)}, \quad \text{all } r \geq 1,$$

then, with this modification taken into account, it follows that $g_r \propto a^0 = 1$, for all relevant r , and the door to fidelity is open. To achieve that uniform rescaling, we closely follow the scheme presented in ref. 1.

ALTERNATIVE LATTICE MODEL

The sought-for generating functional for lattice-space Schwinger functions $S\{h\}$ will be expressed in the form

$$S\{h\} \equiv [T\{h\}]^{N_R},$$

$$T\{h\} \equiv FN \int e^{\sum h_k \varphi_k a^3 - \mathfrak{A} - \mathfrak{B}} \Pi d\varphi_k.$$

Let us examine the ingredients in these expressions separately.

The conventional probability distribution for a φ_3^p model is given by

$$D \equiv N e^{-\mathfrak{A}},$$

where

$$\mathfrak{A} \equiv \frac{1}{2} \sum (\varphi_{k^*} - \varphi_k)^2 a + \frac{1}{2} m_o(a)^2 \sum \varphi_k^2 a^3 + \lambda(a) \sum \varphi_k^p a^3,$$

and

$$N^{-1} \equiv \int e^{-\mathfrak{A}} \Pi d\varphi_k.$$

As before, $k \in \mathbb{Z}^3$ denotes a lattice site, and k^* denotes each of the three positive nearest neighbors to k .

We now modify the distribution D as follows. First, let

$$F \equiv K a^{(p-6)/(p-2)},$$

where K is a positive constant, and focus on the region where $F < 1$. Next, consider

$$FNe^{-\mathfrak{A}},$$

which (due to F) is no longer a normalized distribution. To restore normalization, we introduce an auxiliary, nonclassical (proportional to \hbar^2) term given by

$$\mathfrak{B} \equiv \frac{1}{2} A(a) \Sigma[\varphi_k^2 - B(a)] / [\varphi_k^2 + B(a)]^2 a^3$$

into the action to yield

$$D' \equiv FNe^{-\mathfrak{A} - \mathfrak{B}}.$$

The potential chosen for \mathfrak{B} is a regularized version of $A/2\varphi^2(x)$, which is the only “pure” counterterm that introduces no new dimensional coupling constant for any spacetime dimension. The factors $A(a)$ and $B(a)$ that appear in \mathfrak{B} are positive and, as discussed below, they are chosen so that D' is a normalized distribution.

At this stage all the various correlation functions are small (of order F) and need to be brought back to normal size. To that end, we raise the generating functional $T\{h\}$ to the power $N_R \equiv \llbracket a^{-(p-6)/(p-2)} \rrbracket$, where $\llbracket \cdot \rrbracket$ denotes the integral part of its argument. The result is a new generating functional, $S\{h\}$, all truncated correlation functions of which are increased in magnitude over those of $T\{h\}$ by the factor N_R . One may understand this procedure as allowing for the use of reducible sharp-time field operator representations, a liberalization well known to be important in advanced quantum field theory studies.⁽⁷⁾

As the final step in our construction we take the continuum limit $a \rightarrow 0$, accompanied by another limit in which the lattice size grows to eventually cover all of \mathbb{R}^3 . In the continuum limit we require that $B(a) \rightarrow 0$, while $A(a)$ may or may not diverge as $a \rightarrow 0$. If $A(a)$ diverges we assume that it diverges at a rate slower than a^{-4} (see ref. 1); in that case, the discussion regarding how the form of \mathfrak{B} has been chosen is identical to the discussion presented in ref. 1, and so it is not repeated here. As in ref. 1, after the continuum and infinite volume limits, the final result for $S\{h\}$ corresponds to a generalized Poisson distribution.⁽⁸⁾

It is noteworthy that an analogous construction has been carried out rigorously and successfully in *one* spacetime dimension, i.e., Euclidean time alone; see Chap. 10 in ref. 2. This calculation was not motivated by a study of any of the usual nonrenormalizable theories, but it may be used to lend credence to the present proposal for such models when analyzed by similar methods.

Approximate Evaluation

In addition, in ref. 1, we presented a crude approximation for evaluating the normalization condition that ensures that D' is a probability distribution. That argument can also be carried over directly to the present situation. In this approximate calculation it was assumed that the entire lattice volume $\mathcal{V} = (La)^3$, where L denotes the number of lattice points on one edge of the cubic lattice, is divided into M cells of volume $v = (\xi a)^3$, where ξ denotes an approximate correlation length. For simplicity in evaluation it was assumed that all field variables within a correlation volume v were exactly correlated, while field variables in distinct correlation volumes were assumed to be entirely uncorrelated. Moreover, in any calculation establishing normalization of D' , one first chooses the behavior of $B(a)$ in a suitable way (see ref. 1), e.g., as

$$B(a) = |\ln(a)|^{-2},$$

and then determines $A(a)$ in relation to that choice. The strong simplifications that were made led to a rough, approximate expression for $A(a)$ given by

$$A(a) = (2/M) [(p-6)/(p-2)] (La)^{-3} |\ln(a)|^{-1},$$

Unfortunately, this result for $A(a)$ is only a leading order estimate, which is insensitive to important parameters such as m_0^2 and λ . More precise determination of $A(a)$ would include its dependence on such model parameters, in particular on the coupling constant λ .

PSEUDOFREE THEORY

It is important to note that the factor F which rescales all the correlation functions is *independent* of λ . If we consider the limit of the interacting theory as $\lambda \rightarrow 0^+$, it must be kept in mind that the resultant limit will *not* be the conventional free theory. Instead, the limiting theory is what we call the *pseudofree* theory.^(1,2) The pseudofree theory is therefore the noninteracting theory to which the interacting theory is continuously connected. Stated otherwise, the conventional free theory is not even continuously connected to the interacting theory, and therefore perturbation-theoretic generated counterterms are not reliable! As a consequence, the pseudofree theory acquires interest in its own right, and from a computational point of view it would be a good place to begin because it has one less parameter than the interacting theory. Note as well that each power p in the φ_3^p models seems to correspond to a different pseudofree theory since the parameter p enters

into F and therefore into $A(a)$ in an apparently significant way. It would be of considerable interest if Monte Carlo methods could be used to satisfy the normalization condition and thereby to help determine the pseudofree theory for one or more p values.

QUANTUM FIELDS

We expect all the models discussed in this paper to correspond to quantum field theories after Wick rotation for the following reasons: The essential requirements to lead to a quantum field theory are Euclidean invariance, reflection positivity, moment growth, and clustering. These conditions are satisfied by the original theory and are not disturbed by an overall scaling (F) and an additional local interaction (\mathfrak{F}). Multiple copies also preserve these properties. So long as there is a uniform lower bound on the mass, the continuum and infinite volume limits should have the desired effect.

OTHER MODELS

Although we have confined our attention in this paper to φ_3^p models for values of $p \geq 8$, it should be evident that similar methods can be extended to multicritical points associated with other nonrenormalizable models such as φ_n^p whenever $p > 2n/(n-2)$ and $n \geq 4$. This analysis would therefore extend the class of models considered in ref. 1. The essential changes to major formulas given in this paper would be that in this more general case

$$g_r \propto a^{(r-1)[n(p-2)-2p]/[p-2]},$$

for all relevant r , which would involve a uniform rescaling such that

$$\langle \varphi_{k_1} \varphi_{k_2} \cdots \varphi_{k_r} \rangle \propto a^{[n(p-2)-2p]/[p-2]}, \quad \text{all } r \geq 1.$$

To obtain this rescaling requires that

$$F = Ka^{[n(p-2)-2p]/[p-2]},$$

and, correspondingly, that

$$N_R = \llbracket a^{-[n(p-2)-2p]/[p-2]} \rrbracket.$$

With these changes, the discussion is substantially similar to that given in the present paper, augmented when necessary by the contents of ref. 1.

DEDICATION

It is a pleasure to dedicate this article to the 70th birthday of Elliott Lieb. Elliott is a long-time friend and someone I have admired for his analytical skills and his remarkable originality for many years. I wish him a long life, full of happiness and continued quality research.

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